Class notes for 30 September

Reminders
- Problem Set 3 due Oct. 7 (9.7, 9.11, 9.27, 10.9, 10.13, 10.15)
- Exercise 3 due date postponed until Oct. 5

10.1 Hydrostatic Equilibrium

Big ideas
- We can study the outsides of stars, but can't see the insides directly
- Physics is known (sort of...), need computers and codes.
- Stellar evolution driven by gravity, contraction opposed by pressure.
- Hydrostatic equilibrium is the balance between pressure and gravity.

Consider a small mass $dm$ located at a distance $r$ from the center of a star. The force of gravity, directed downward, is

$$F_g = -G \frac{M_r dm}{r^2}$$

The force due to pressure, directed upward, is the difference in the pressure on the bottom surface of $dm$ compared to the top surface ($dF_p = AdP$).

Recall that density is $\rho = \frac{dm}{Adr}$ (mass per unit volume).

If the star is static - the gas isn't moving, then

$$\frac{dP}{dr} = -G \frac{M_r \rho}{r^2}$$

or

$$\frac{dP}{dr} = -\rho g$$

where $g = GM_r/r^2$, the gravitational acceleration.

This is known as the equation of hydrostatic equilibrium.

What is the pressure at the center of the Sun?

Assume $M_r = 1 \ M_{\text{Sun}}$, $r = 1 \ R_{\text{Sun}}$, $\rho = \text{average solar density} = 1.41 \ g \ cm^{-3}$.
Assume the pressure at the surface is zero.

Then $dP/dr = -P_{\text{center}}/R_{\text{Sun}} = \rho g$, $P_{\text{center}} = \pi g R_{\text{Sun}} = 2.7 \times 10^{15} \ \text{dynes cm}^{-2}$.

A more rigorous calculation (integrating the eqn of hydrostatic equilibrium) gives $2.5 \times 10^{17} \ \text{dynes cm}^{-2}$. (One atmosphere of pressure is $1 \times 10^6 \ \text{dynes cm}^{-2}$).

Stars must also satisfy the mass conservation equation: $dM_r/dr = 4\pi r^2 \rho$. 

10.2 Pressure Equation of State

The equation of state describes how pressure varies with other fundamental parameters (for example, temperature and density).

Remember the **ideal gas law**: \( PV = NkT \)? This is an equation of state. The ideal gas law is often useful as the equation of state in astrophysics, but there are also many environments where a different equation of state is needed.

The form of the ideal gas law usually used in astronomy uses the concept of the mean molecular weight, defined as

\[
\mu = \frac{\bar{m}}{m_H},
\]

where the mean molecular weight is the average mass of a free particle, including electrons, atoms, and ions. Thus the mean molecular weight depends on the ionization state of the gas. Compute it by summing the weight of each particle times the number of that particle, divided by the total number of particles.

Using the mean molecular weight, we can redefine the ideal gas law as

\[
P_g = \frac{\rho k T}{\mu m_H}
\]

The concept of mass fractions is also useful. We define the three mass fractions as

\[
X = \text{total mass of hydrogen} / \text{total mass of gas}
\]

\[
Y = \text{total mass of helium} / \text{total mass of gas}
\]

\[
A = \text{total mass of all other atoms} / \text{total mass of gas}
\]

For a completely neutral gas, \( \frac{1}{\mu_n} = X + \frac{1}{4}Y + \left(\frac{1}{A}\right)_n Z \)

For a completely ionized gas, \( \frac{1}{\mu_i} = 2X + \frac{3}{4}Y + \left(\frac{1+z}{A}\right)_n Z \)

For \( X=0.70, Y=0.28, \) and \( Z=0.02, \) then \( \mu_n = 1.3- \) and \( \mu_i = 0.62. \)

The ideal gas law does not hold in all circumstances. Particle speeds cannot exceed the speed of light, and quantum mechanics also leads to deviations (i.e. in extremely dense matter. Radiation pressure can also play a role, and sometimes can exceed the gas pressure. If radiation pressure exceeds gravity, the star can expand.
Estimate the central temperature of the Sun:

Neglecting radiation pressure, the central temperature of the Sun is given by the ideal gas law:

\[ T_c = \frac{P_c \mu_m \mu_i}{\rho k} \]

Using the mean density, and \( \mu_i = 0.62 \) (for a completely ionized gas), and the estimated value for the central pressure (\( P_c = 2.7 \times 10^{15} \text{ dynes cm}^{-2} \)), we get a central temperature

\[ T_c = 1.44 \times 10^7 \text{ K} \]

A standard solar model gives \( 1.58 \times 10^7 \text{ K} \).

At this temperature and pressure, radiation pressure is only 0.06\% of the gas pressure.